Received: *March 29, 2000*, Accepted: *April 25, 2000*

# Schwinger-Dyson = Wheeler-DeWitt: gauge theory observables as bulk operators

**Gilad Lifschytz and Vipul Periwal**

*Department of Physics, Joseph Henry Laboratories*
*Princeton University Princeton, NJ 08544, USA*
*E-mail:* gilad@viper.princeton.edu, vipul@princeton.edu

Abstract: We argue that the second-order gauge-invariant Schwinger-Dyson operator of a gauge theory is the Wheeler-DeWitt operator in the dual string theory. Using this identification, we construct a set of operators in the gauge theory that correspond to excitations of gravity in the bulk. We show that these gauge theory operators have the expected properties for describing the semi-classical local gravity theory.

Keywords: D-branes, String Duality, Models of Quantum Gravity.

JHEP04(2000)026

## Contents





## 1. Introduction

Following Maldacena's conjecture [1, 2, 3], a fair amount of evidence has accumulated in support of the identification of certain (supersymmetric) gauge theories in the large-$N$ limit with (super)gravity theories on appropriate backgrounds with one additional non-compact direction. The existence of a semi-classical (super)gravity background is predicated on taking the 't Hooft coupling $g^2_{\rm YM}N \equiv \lambda$ to be very large, which makes any direct quantitative verification of the conjecture difficult.

If large-$N$, large-$\lambda$ gauge theories are dual to classical gravity backgrounds, we should be able to construct bulk operators in the gravity theory, for example to study the properties of singularities. However, correlation functions of the gauge theory are boundary correlations in the gravity theory, and these are believed to be the only observables. The obvious question that needs to be addressed then is: how is bulk information encoded in boundary correlation functions? In fact we do not yet understand well enough even the relationship between classical gravity concepts and the dual gauge theory, although some understanding on issues like causality and horizons have been gained [5]. While the identification of the extra large dimension with some energy scale [1] has enabled one to deduce certain properties of objects in the bulk as seen from the gauge theory, a map between local bulk operators and gauge theory observables has not been found. The case of a classical non-interacting field on the AdS background was discussed in [6].

This is intimately related to an intriguing aspect of AdS/CFT duality, the idea of holography, first proposed by 't Hooft [7]. Briefly, black hole thermodynamics suggests that the degrees of freedom inside the horizon of a black hole reside at the horizon. This is difficult to reconcile with the extensivity that we expect in garden-variety low-energy effective field theories. 't Hooft and Susskind [7] suggested that this behaviour might be a general feature of quantum gravity. In fact, the AdS/CFT duality turns out to be an explicit example of holography [3, 8]: the (super)gravity theory is defined in five non-compact dimensions for four-dimensional gauge theory.

Some time ago, Ishibashi, Kawai and their collaborators [9] constructed toy models of quantum gravity coupled to $c < 1$ matter in temporal gauge. It turned out that the string field theories they constructed were related to matrix models [10] via the stochastic quantization of Parisi and Wu [11]. Extrapolating these ideas to gauge theories using the observation of Marchesini [12] led to a proposal for a direct non-perturbative connection between gauge theories and string theories in temporal gauge [13], making contact with Polyakov's ideas regarding gauge theories and non-critical strings [14].

We consider in this paper the proposition that the second-order gauge-invariant Schwinger-Dyson equations of the gauge theory are the Wheeler-DeWitt equations of string theory, as Ishibashi et al. [9] did for the $c < 1$ models. We explain why such an identification depends crucially on the structure of the Schwinger-Dyson equations expressed in terms of Wilson loops. This identification allows us to construct gauge theory observables that are naturally related to operators in the bulk of the dual gravity background, shedding some light on the manner in which holography is implemented in the AdS/CFT duality. We show that these operators have the expected properties, using the fact that the second-order Schwinger-Dyson equations are in fact just the equilibrium conditions of stochastic quantization, with the operator that generates the Schwinger-Dyson equations also the Fokker-Planck hamiltonian of the gauge theory. This connection has already been used to suggest a connection between the radial direction of AdS and stochastic time, and to explain the finite $N$ truncation of the spectrum of chiral primary operators in the gauge theory [16] from the string theory perspective. Some related work can be found in [17].

The plan of this paper is as follows: we start with a brief account of the relevant results in the stochastic quantization of gauge theories. We then motivate an identification of the Schwinger-Dyson operator of the gauge theory with the Wheeler-DeWitt operator of the gravity theory. Using this identification a set of "bulk" operators is constructed from the gauge theory and their properties explored.

## 2. Stochastic quantization

Stochastic quantization [11, 18] is a representation of quantum correlation functions as equilibrium values of correlation functions of a classical system coupled to white noise.

Given a classical equation of motion for an euclidean field theory in $d$ dimensions, we associate a Langevin equation

$$\frac{\partial\phi}{\partial\tau} = -\Omega\,\frac{\delta S}{\delta\phi} + \eta\,(x,\tau)\,. \tag{2.1}$$

where $x$ is a point in $d$ dimensions, $\tau$ is a fiducial euclidean "time" coordinate, $\Omega$ is a time scale, and $\eta$ is white noise with a noise ensemble average

$$\Big\langle\eta\,(x,\tau)\eta\,(x',\tau')\Big\rangle_\eta = \Omega\,\delta\,(x-x')\,\delta\,(\tau-\tau')\,. \tag{2.2}$$

(We will set $\Omega = 1$ for most of the paper.) The basic statement of stochastic quantization is that the equal $\tau$ equilibrium stochastic correlation functions are equal to the quantum correlation functions of the original theory: if $\phi_\eta$ is the solution of the Langevin equation (2.1) with some initial condition then

$$\lim_{\tau\uparrow\infty}\Big\langle\prod_i\phi_\eta(x_i,\tau)\Big\rangle_\eta = \Big\langle\prod_i\phi(x_i)\Big\rangle\,, \tag{2.3}$$

where the left-hand side is the stochastic average, which is independent of the initial condition in the large-$\tau$ limit, and the right-hand side is the vacuum correlation function in the quantum field theory. Equal $\tau$ equilibrium ($\tau\uparrow\infty$) correlation functions are, of course, just a particular case of more general unequal $\tau$ correlation functions:

$$\Big\langle\phi_\eta\,(x_1,\tau_1)\cdots\phi_\eta\,(x_n,\tau_n)\Big\rangle_\eta \equiv \int d\eta\,(x,t)\,e^{-\int\eta^2/\Omega}\,\phi_\eta\,(x_1,\tau_1)\cdots\phi_\eta\,(x_n,\tau_n)\,. \tag{2.4}$$

An equivalent formulation of stochastic quantization is obtained by finding a stochastic action $S_{\rm stoc}[\phi(x,\tau)]$ such that the correlation functions computed in a functional integral with this action are the stochastic correlation functions. For a scalar field theory

$$S_{\rm stoc} = \int d\tau\,dx\left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{8}\left(\frac{\delta S}{\delta\phi}\right)^2 - \frac{1}{4}\left(\frac{\delta^2 S}{\delta\phi(x)^2}\right)\right]. \tag{2.5}$$

The last term is divergent and must be understood in the context of a regularization. In dimensional regularization, it is set to zero as usual, but this choice of regularization may not be appropriate in all instances. Associated with this action is a hamiltonian which generates translations in the $\tau$ direction. This is the Fokker-Planck (FP) hamiltonian

$$H_{\rm FP} = \int dx\left[\frac{\delta}{\delta\phi(x)} - \frac{\delta S}{\delta\phi(x)}\right]\frac{\delta}{\delta\phi(x)}\,. \tag{2.6}$$

This hamiltonian is not hermitean but becomes hermitean under a similarity transformation by $e^{-S/2}$

$$\hat{H}_{\rm FP} = \int dx \left[ -\frac{1}{2}\frac{\delta^2}{\delta\phi^2} + \frac{1}{8}\left(\frac{\delta S}{\delta\phi}\right)^2 - \frac{1}{4}\frac{\delta^2 S}{\delta\phi^2} \right]. \tag{2.7}$$

Introducing a source $J$ for $\phi$, we can define a second-order Schwinger-Dyson (SD) operator $H_{\rm SD}$ by

$$\left\langle H_{\rm FP}\, e^{\int J\phi} \right\rangle \equiv H_{\rm SD}\left(J, \frac{\delta}{\delta J}\right) \left\langle e^{\int J\phi} \right\rangle, \tag{2.8}$$

where $\langle\cdots\rangle$ denotes the expectation value in the field theory. The equal $\tau$ correlation functions can also be represented as (we have set $\Omega = 1$)

$$\left\langle \prod_i \phi_\eta(x_i, \tau) \right\rangle_\eta = \left\langle\!\left\langle 0 | e^{-\tau H_{\rm FP}} \phi(x_1)\cdots\phi(x_n) | 0 \right\rangle\!\right\rangle, \tag{2.9}$$

where in the large-$\tau$ limit one gets the correlation functions of the original field theory, and we can use either $H_{\rm FP}$ or $\hat{H}_{\rm FP}$. The existence of the large-$\tau$ limit implies

$$\left\langle\!\left\langle 0 | e^{-\tau H_{\rm FP}} H_{\rm FP} \phi(x_1)\cdots\phi(x_n) | 0 \right\rangle\!\right\rangle = 0\,; \tag{2.10}$$

these are the equilibrium conditions of stochastic quantization. The expectation value in eq. (2.9) is defined with respect to a formal vacuum state (which depends on whether one uses $H_{\rm FP}$ or $\hat{H}_{\rm FP}$) satisfying

$$\left\langle\!\left\langle 0 \right| H_{\rm FP}^\dagger = 0\,, \tag{2.11}$$

$$H_{\rm FP} \left| 0 \right\rangle\!\right\rangle = 0\,. \tag{2.12}$$

We shall be interested in another set of correlation functions, those in which the stochastic time is taken to infinity but the correlation functions depend on finite differences of stochastic time between the operators:

$$\lim_{\tau\to\infty} \left\langle \mathcal{O}_1(x_1, \tau + t_1)\mathcal{O}_2(x_2, \tau + t_2)\cdots\mathcal{O}_n(x_n, \tau) \right\rangle_\eta. \tag{2.13}$$

While finite $\tau$ correlation functions cannot be directly interpreted as quantum correlations, the correlation functions in eq. (2.13) are in fact quantum correlations in the original theory. They contain no new information of course since they can be written as a sum of equal time correlation functions with coefficients depending on $t_i - t_j$. These correlations are translation invariant in $t_i$.

Turning to gauge theories, the Fokker-Planck hamiltonian for gauge theories

$$\int dx\, \frac{1}{N} \sum_{\mu,a} \left( \frac{\delta}{\delta A_\mu^a(x)} - \frac{\delta S}{\delta A_\mu^a(x)} \right) \frac{\delta}{\delta A^{\mu a}(x)} \tag{2.14}$$

has the property that it is gauge invariant. Acting on Wilson loops, the action of this operator (or of its associated Schwinger-Dyson operator) can be interpreted as Wilson loop diffusion, joining and splitting processes.

A physical interpretation of stochastic time is obtained by introducing Schwinger proper-time representations of propagators. Viewing all Feynman diagrams as starting from external legs at $\tau = 0$, the time evolution is generated by the Fokker-Planck hamiltonian [19]. In this interpretation, a finite $\tau$ amplitude is a sum over Feynman diagrams with the restriction that the total proper-time is bounded. Thus finite $\tau$ amplitudes are infrared regulated. For example, for a free massless scalar field the two-point function at finite stochastic time (with appropriate boundary conditions) is

$$\Big\langle \phi_\eta(k,\tau)\, \phi_\eta(-k,\tau) \Big\rangle_\eta = \frac{1}{k^2}\Big(1 - e^{-k^2\tau}\Big) = \int_0^\tau ds\ e^{-sk^2} \tag{2.15}$$

which exhibits no singularity as $k^2 \downarrow 0$.

### 2.1 The Schwinger-Dyson equations

In gauge theories, there is a natural complete set of gauge-invariant operators. These are Wilson lines $\mathcal{O}_C \equiv \mathrm{Tr}\, P e^{i \oint_C A}$ with Tr a normalized trace defined by $\mathrm{Tr}(1) = 1$. This string (or contour) labeling of the operators leads to an important feature of the Schwinger-Dyson operator in gauge theories: it is *second-order* in functional derivatives with respect to sources for single Wilson loop operators. This is crucial in interpreting the Schwinger-Dyson operator as the Wheeler-DeWitt operator. Let us consider this in some detail: the point is that the term $\delta^2 \mathcal{O}_C/\delta A^2$ in eq. (2.14) for Wilson loops gives

$$\begin{aligned}\int dx \sum_{\mu,a} \frac{\delta^2}{\delta A_{\mu a}(x)^2} \mathrm{Tr}\ \mathrm{P} e^{i\oint A} = -\sum_a \ & \oint ds_1 ds_2 \Big(\dot{x}(s_1)\cdot \dot{x}(s_2)\Big) \times \\ & \times \mathrm{Tr}\ \mathrm{P}\left( e^{\int_{x(s_1)}^{x(s_2)} A} T_a e^{\int_{x(s_2)}^{x(s_1)} A} T_a \right) \times \\ & \times \delta\Big(x(s_1) - x(s_2)\Big)\,. \end{aligned} \tag{2.16}$$

Using the identity

$$\sum_a \mathrm{Tr}\,(T_a X T_a Y) = N \cdot \mathrm{Tr}\, X \cdot \mathrm{Tr}\, Y \tag{2.17}$$

valid for $\mathrm{U}(N)$ matrices, with $T_a$ a basis for normalized hermitean matrices, this splits a Wilson loop into two Wilson loops. Thus, the Schwinger-Dyson equations in a gauge theory are schematically

$$\begin{aligned} H_{\mathrm{SD}}\, e^{W[J]} \equiv \sum_C \Bigg[ J^C \Bigg( & \sum_{C',C'':(C'C'')=C} \frac{\delta}{\delta J^{C'}} \frac{\delta}{\delta J^{C''}} - \frac{1}{\lambda}\frac{\delta}{\delta J^{\hat{C}}} \Bigg) + \\ & + \frac{1}{N^2} \sum_{C'} J^C J^{C'} \frac{\delta}{\delta J^{(CC')}} \Bigg] e^{W[J]} = 0\,. \end{aligned} \tag{2.18}$$

Here we have defined $(AB)$ as the contour obtained by joining contours $A$ and $B$ when they have a point in common, and $\hat{C}$ is the contour obtained by joining the "infinitesimal" contour associated with the action $S$ into the contour $C$. We digress briefly to make precise the notion of an "infinitesimal" contour.

It is important to consider the regularization of these equations (2.18). Wilson loops are composite operators and even in a finite theory such as $N = 4$ supersymmetric gauge theory, correlation functions of composite operators must be defined with a regularization and renormalization scheme. In particular, the action of a regularized gauge theory can be interpreted as a sum of Wilson loops of size equal to the regularization length scale, as in lattice gauge theory, and it is this interpretation that is the definition of an "infinitesimal" contour. The Schwinger-Dyson equations (2.18) must be regularized as well, but happily a gauge invariant regularization procedure has been developed for these [20]. The fact that a regularization must be introduced at this stage implies that there will be a normalization scale implicit in the renormalized equations. It may be that this normalization scale is related to the string tension of type-IIB string theory.

Let us compare the structure of the SD equations for gauge theories with the same structure for scalar field theory. Given a complete set of local operators $\mathcal{O}_i$ in a scalar field theory, we can define the generating function

$$Z[J] \equiv e^{W[J^i]} \equiv \left\langle e^{\sum_i J^i \mathcal{O}_i} \right\rangle . \tag{2.19}$$

If the scalar field theory is defined by an action $S$, we can derive a functional differential equation satisfied by $e^{W[J]}$:

$$H_{\mathrm{SD}} e^{W(J^i)} \equiv \sum_i \left[ J^i \left( \frac{\delta}{\delta J^{i''}} - \frac{\delta}{\delta J^{\hat{\imath}}} \right) + \sum_k J^i J^k \frac{\delta}{\delta J^{(ik)}} \right] e^{W(J^i)} = 0 \tag{2.20}$$

with $i''$ the index associated with the operator $\delta^2 \mathcal{O}_i / \delta\phi^2$, $\hat{\imath}$ the index associated with the operator $(\delta\mathcal{O}_i/\delta\phi)(\delta S/\delta\phi)$, and $(ik)$ the index associated with the operator $(\delta\mathcal{O}_i/\delta\phi(x))(\delta\mathcal{O}_k/\delta\phi(x))$. Thus it is not necessary to introduce a second-functional derivative with respect to sources in the SD equations for a scalar field theory.

It is a non-trivial fact that the second-order Schwinger-Dyson equations (2.18) are the equilibrium conditions of stochastic quantization in eq. (2.10). In gauge theories, the second-order Schwinger-Dyson equations [12] expressed in terms of Wilson loops are just the so-called loop equations [15].

## 3. Relationship between SQ and AdS/CFT

According to the AdS/CFT duality, the partition function of the gauge theory coupled to sources is the exponential of the supergravity action evaluated as a function of the boundary values of the supergravity fields. The duality holds at large $N$ and large $\lambda$ for slowly varying configurations [2, 3]:

$$e^{W[J]} \equiv \left\langle e^{\int J\mathcal{O}} \right\rangle_{\rm CFT} = e^{-S_{\rm sugra}[\hat{J}(J)]} \,. \tag{3.1}$$

We have written the supergravity boundary values as $\hat{J} = \hat{J}(J)$ to indicate that rescalings may be needed depending on the dimension of the operator $\mathcal{O}$.

The right-hand side of eq. (3.1) can be interpreted as the leading order term in a path integral representation $\Psi[\hat{J}]$ of a wave functional of supergravity, where the class of metrics we integrate over has one boundary, i.e. it is the no-boundary wave function of Hartle and Hawking. This wave functional $\Psi[\hat{J}] \approx e^{-S_{\rm sugra}[\hat{J}(J)]}$ obeys the Wheeler-DeWitt equation

$$H_{\rm WD}\Psi[\hat{J}] \approx 0 \,, \tag{3.2}$$

where the Wheeler-DeWitt operator $H_{\rm WD}$ is just the supergravity hamiltonian, the sum of the gravity and matter hamiltonians.

Before continuing let us consider exactly how one gets to the supergravity limit from the gauge theory. The generating function $W[J]$ of the gauge theory is a function of all the possible sources. We separate these into sources for supergravity excitations $J_{\rm sugra}$ and sources for the rest which we will call string sources $J_{\rm string}$. In order to obtain the effective supergravity description one does not simply set $J_{\rm string} = 0$. The correct prescription is to solve for $J_{\rm string} = J_{\rm string}(J_{\rm sugra})$, in other words to solve for massive backgrounds in terms of slowly varying massless backgrounds. The requirement that there is no production of string excitations in the effective supergravity theory in any low-energy supergravity scattering, translated into conditions in terms of the gauge theory operators that couple to either supergravity fields or to string fields, is

$$\left\langle \mathcal{O}^1_{\rm sugra} \cdots \mathcal{O}^n_{\rm sugra}\mathcal{O}_{\rm string} \right\rangle = 0 \,. \tag{3.3}$$

In terms of $W[J]$ this is

$$\frac{\delta W[J]}{\delta J_{\rm string}} = 0 \,, \tag{3.4}$$

which is the usual procedure of solving the equations of motion of massive backgrounds as functions of massless backgrounds. This in fact is exactly how the supergravity approximation is supposed to arise from a string field theory. The non-linear terms in the Einstein action arise only after integrating out the massive string modes.

From the basic identification of the AdS/CFT duality equation (3.1), notice that if we find a solution of either eq. (2.18) or eq. (3.2) (with appropriate boundary conditions), we solve the whole theory. Both equations depend only on boundary values of bulk fields, and both equations are second-order in functional derivatives. We emphasize again that the second-order form of the SD operator is a consequence of the natural operators in the gauge theory being Wilson loops. An identification of the SD operator with the WD operator in the supergravity limit is therefore indicated.



JHEP04(2000)026

Beyond the supergravity limit, the supergravity hamiltonian (the WD operator) must be replaced by the complete string field theory hamiltonian. Thus the SD operator of the gauge theory, written in terms of its action on Wilson loops, should be identified with the string field theory hamiltonian. Indeed, the structure of the SD operator that follows from its derivation from the FP hamiltonian has exactly the appropriate form to be a string field theory hamiltonian since the FP hamiltonian takes the form

$$H_{\mathrm{FP}} \sim \frac{1}{\lambda}\,[\text{Diffusion} + \text{Tadpole}] + [\text{Loop splitting}] + \frac{1}{N^2}\,[\text{Loop joining}]\,. \tag{3.5}$$

The identification of the WD operator with the SD operator suggests a relation between the radial coordinate in AdS and the stochastic time coordinate in stochastic quantization [16]. For example, let us see how the computation of a Wilson loop is viewed in stochastic quantization. We start with

$$\left\langle\!\left\langle 0|e^{-\tau H_{\mathrm{FP}}}W(C)|0\right\rangle\!\right\rangle. \tag{3.6}$$

For a smooth non-intersecting loop, the Fokker-Planck operator deforms the loop. In eq. (3.6), we are calculating the amplitude for the loop to disappear into the vacuum. In the large 't Hooft coupling limit ($\lambda \uparrow \infty$) the most economical way is to continuously deform the loop to zero size where it is annihilated by the tadpole operator. If we write eq. (3.6) as a functional integral, the path (in loop space) that the loop takes to approach zero should then be a saddle point of the action for large $\lambda$. This is exactly like the picture of the Feynman graphs at the end of section 2, and is also how the computation proceeds in the AdS/CFT correpondence where the loop's path in loop space is computed using the minimal area in the AdS background [21]. Thus evolution in the stochastic time direction is like evolving from the boundary of AdS inwards. As such the evolution operators on both sides should be identified up to the sign resulting from the direction of evolution, inwards (FP) or outwards (WD). As the AdS side correpond to looking at sources for the Wilson loops, it is the SD operator that becomes the operator that gives the classical string equations in the background. Of course, loops are deformed in exactly the same manner for either the Wilson loop or its source.

We hasten to add that the above description does not imply directly that evolution inwards from the boundary in anti-de Sitter space is directly related to the stochastic evolution. We must keep in mind that the gauge theory is recovered in the limit $\tau \uparrow \infty$. As will become clear in the next section, even taking $\tau \uparrow \infty$ leaves a set of correlation functions that have operators separated in stochastic time.

## 4. Bulk operators

In semi-classical gravity there is a notion of approximate locality in spacetime. We should be able to calculate properties of processes that are approximately localized in the bulk. We would like to construct a set of operators in the gauge theory that reflects this property. It has been suggested that the radial direction corresponds to a cut-off scale or renormalization scale, but what we need is a set of operators that can be defined at any point in spacetime. These will then give correlation functions at any $n$ distinct points in the bulk. Let us label by $\mathcal{O}_i(0)$ the set of gauge theory operators whose correlation functions are given by varying the string theory wave functional with respect to the boundary value of the string fields. In the supergravity limit these are just the local gauge invariant operators. While this set of operators is a complete set in the gauge theory, its interpretation in the gravity theory seems to confine it to a hypersurface in spacetime. The string theory is holographic, i.e. all its observables are those of the gauge theory. In the low-energy semi-classical limit, we should still be able to find observables of the gauge theory that can be interpreted as the observables of a theory in one dimension higher.

Using the understanding we have developed in previous sections of how the bulk arises in the gauge theory viewed via stochastic quantization, it is easy to see that there is a set of operators with the appropriate evolution in the radial direction. As we have discussed, the SD operator, in the semi-classical limit, is the generator of translations in the radial direction. The SD operator does not act directly on the gauge theory operators $\mathcal{O}_i$. From eqs. (2.8) and (2.13), we see that an appropriate operator can be defined as

$$\mathcal{O}(t) \equiv e^{-tH_{\mathrm{FP}}}\mathcal{O}(0)e^{tH_{\mathrm{FP}}} \,. \tag{4.1}$$

The right-hand side of eq. (4.1) is an operator in the original gauge theory (although a rather unusual one). We claim that gauge theory correlation functions with operators as in eq. (4.1) convey the information of the bulk theory. Thus we will call these bulk operators. We would like to understand the properties of these operators and the connection between the parameters $t_i$ and the physical radial distance in the AdS.

Let us look at the equation of motion of the expectation value of a local bulk operator in the gauge theory

$$f[J] \equiv \left\langle \mathcal{O}(x,t)e^{\int J\mathcal{O}}\right\rangle , \tag{4.2}$$

with $\langle\cdots\rangle$ the gauge theory vacuum expectation value, as before. The equation of motion is then

$$\frac{d}{dt}f[J] = -\left\langle [H_{\mathrm{FP}}, \mathcal{O}(x,t)]e^{\int \mathcal{O}J}\right\rangle . \tag{4.3}$$

Let us define $\Pi_J(x) \equiv \delta/\delta J(x)$, then the above equation can be recast in terms of sources to be

$$\frac{d}{d\tilde{t}}\,\Pi_J(x,\tilde{t})\left\langle e^{J\mathcal{O}}\right\rangle = -\Big[H_{\mathrm{SD}}, \Pi_J(x,\tilde{t})\Big]\left\langle e^{\int J\mathcal{O}}\right\rangle , \tag{4.4}$$

where we have defined $\Pi_J(x,\tilde{t}) = e^{-\tilde{t}H_{\rm SD}}\Pi_J(x)e^{\tilde{t}H_{\rm SD}}$, and $\tilde{t} = -t$. The minus sign just reflects that the natural parametrization from the gravity point of view is toward the boundary of AdS while from the gauge theory it is inwards. Together with eq. (4.4) that we will rewrite as

$$\frac{d}{d\tilde{t}}\,\Pi_J(x,\tilde{t}\,) = -\Big[H_{\rm SD}, \Pi_J(x,\tilde{t}\,)\Big]\,, \tag{4.5}$$

one also has the equation

$$H_{\rm SD} = 0\,. \tag{4.6}$$

Thus we see that if we identify the SD operator in the gravity limit (which in the gauge theory was described in section 3) with the WD operator, then eqs. (4.5) and (4.6) is just the classical equation of motion of the gravity theory. What we are missing is the four-dimensional diffeomorphism constraint, but it follows just from the fact that $J(x)$ is a parameter in the gauge theory integral. So our bulk operators defined above obey the classical gravity equation of motion. Conversely, since the SD operator in the semi-classical limit is hard to compute, we can start instead with the WD operator. The WD operator written in terms of supergravity fields and momenta (sources $J$ and $\Pi_J$) can be rewritten as an operator in the gauge theory variables ($\mathcal{O}$ and $\delta/\delta\mathcal{O}$). If we now use this operator (which is easy to compute starting from the supergravity) to construct the gauge theory bulk operators as in eq. (4.1), we are guaranteed to obtain operators which obey the supergravity equations of motion.

Let us examine in more details how the semi-classical limit will emerge. The label $t$ which we have been using is not the physical radial distance, for any semi-classical spacetime. The semi-classical limit is the one in which we first solve for the classical part of the metric (in the AdS case this is just the conformal part of the metric). The value of the metric plays the role of the physical radial distance. This then can be substituted back into the equation for the fluctuations about the background in order to get the classical evolution of the fields on a fixed spacetime. The parameter $t$ that we have introduced is similar to the foliation label one introduces when quantizing gravity, so the wave function is independent of this parameter. Only after we have solved the metric equations and defined physical time can we connect the parameter $t$ to the radial direction of the AdS. This is evident for instance from the property of translation in $t$ of the bulk correlation functions.

Once this is done however the bulk correlation functions will be identified with local bulk information at various bulk positions. Thus any properties of the correlation functions as a function of the $t_i$ are translated into properties of supergravity correlation functions in the bulk. An example of this was just given above in terms of the equations of motion. However our definition of the bulk operators is valid beyond the classical limit.

In the AdS/CFT connection there is some evidence for a relationship between the RG flow and propagation in the radial direction. A relation between stochas-



JHEP04(2000)026

tic time and the renormalization group is evident from the identification of finite stochastic time $\tau$ amplitudes (eq. (2.9)) as infrared regulated amplitudes [19] (see, e.g. eq. (2.15)). To further our understanding let us look at the RG equation for the $(d+1)$-dimensional stochastic field theory $S_{\rm stoc}$ defined in section 2. Until now we have been relatively naïve, ignoring the issue of renormalization. Stochastic Ward identities restrict the possible renormalization functions of the stochastic theory to be those of the original theory plus renormalization of the stochastic time scale [22], which we have labeled $\Omega$ in the Langevin equation (2.1).

Thus if we introduce a renormalization scale $\mu$, we have [22]

$$\mu \frac{\partial}{\partial \mu} \log \Omega = \eta_\omega (g) \,. \tag{4.7}$$

The renormalization group equation for the Green functions of the stochastic theory is then

$$\left[ \mathrm{RG}_d + \eta_\omega(g)\, \Omega \frac{\partial}{\partial \Omega} \right] \Gamma^n(x_i, t_i; \mu, \Omega) = 0 \,, \tag{4.8}$$

where $\mathrm{RG}_d$ stands for the renormalization group operator in the $d$-dimensional gauge theory. For a conformal field theory (for which the couplings do not run) one has a solution ($\eta_\omega(g)$ is just a number)

$$\frac{\Omega}{\Omega_0} = \left( \frac{\mu}{\mu_0} \right)^{\eta_\omega(g)} \,, \tag{4.9}$$

i.e. if $\mu$ is rescaled so is $\Omega$. Since $\Omega$ is just the scale for the parameter $t$ that is related to the radial direction in the semi-classical limit as explained above, this establishes a connection between the radial direction and the RG flow.

Attempts to directly link supergravity equations of motion to the renormalization group can be found in [23]; see also [24].

Another property of objects in the bulk in the AdS/CFT relationship is that objects in the interior look non local from the perspective of the boundary theory. To see how this arises from the construction of the bulk operators let us look at the free scalar field $\phi$ in four dimensions. The simplest example is to look at the two point function of the operators correponding to those in eq. (4.1):

$$\Big\langle \phi(k,t) \phi(-k,0) \Big\rangle = \frac{1}{k^2}\, e^{-k^2 t} \,, \tag{4.10}$$

or in the space representation,

$$\Big\langle \phi(x,t) \phi(x',0) \Big\rangle \sim \frac{1}{|x-x'|^2} \left[ 1 - e^{-|x-x'|^2/4t} \right] . \tag{4.11}$$

We see that compared with a two point function of two operators at the same $t$ the two point function of operators at differing values of $t$ has an effective UV cut-off. While this example was in the free field case, this property still holds when interactions are turned on. This is exactly in accord with what one expects in the AdS/CFT, but here we see the details of how it works.

This property however should not be confused with the locality of these operators in the bulk. From the supergravity perspective this is a confusing point. If we assume that an operator in the bulk is non local with some non-local scale related to the position in the bulk, then how is it that the supergravity still looks local on scales which are smaller than this non-local scale? Indeed in the AdS/CFT using the metric

$$ds^2 = l_s^2 \left[ \frac{U^2}{\sqrt{2}\lambda}(dx^\nu dx_\nu) + \frac{\sqrt{2\lambda}}{U^2} dU^2 + \sqrt{2\lambda} d\Omega_5^2 \right], \tag{4.12}$$

objects at coordinate $U$ are assumed to have a non-local scale [25] $\sqrt{\lambda}/U$, but supergravity should be valid down to the string scale which is $\sqrt[4]{\lambda}/U$ which is much less than the non-local scale for large 't Hooft coupling $\lambda$. Using the bulk operators this is easy to understand. If we look at some correlation function

$$\Big\langle \mathcal{O}_3(x_3, t_3)\mathcal{O}_2(x_2, t_2)\mathcal{O}_1(x_1, 0) \Big\rangle , \tag{4.13}$$

it has no singularities as the $x_i$ coincide, but if $t_2 \sim t_3 \neq 0$ the correlation function will have singularities when $x_2$ and $x_3$ are close together (i.e. $\mathcal{O}_2$ and $\mathcal{O}_3$ behave as local operators with respect to each other), while with respect to $\mathcal{O}_1$ they behave as non-local operators.

If we assemble some local excitations in the bulk all at some $t_i > t_0$, and probe them with local probes in the gauge theory that are at $t = 0$, eq. (4.10) tells us that from the point of view of the local operators of the gauge theory, the operators in the bulk are only correlated with the low momentum modes of operators at $t = 0$. This is as it should because a bounded region inside the bulk can only support a finite entropy. This is of course related to holography. If we take some UV cut-off for the gauge theory, then there are a finite number of degrees of freedom (for finite $N$), and of course a finite number of independent approximately local operators in the gauge theory. Using the bulk operators we can construct matter distributions inside the anti-de Sitter space, and try to exceed the Bekenstein bound. All the bulk operators can be written as sums of the approximately local operators in the gauge theory, and therefore the number of independent bulk operators cannot exceed the number of degrees of freedom of the original gauge theory.

Another benefit of our construction of bulk operators is the understanding of cluster decomposition in the bulk: why are correlation functions of operators at very different radial positions ($t$) and similar transverse positions ($x$) suppressed? For different transverse positions this is because of the locality of the original theory, but $t$ is not a coordinate in the original space. However the evolution in the $t$ direction is generated by the Fokker-Planck hamiltonian. As we have seen in section 2 regarding the Langevin equation (2.1), the Fokker-Planck hamiltonian generates an



JHEP04(2000)026

approach to equilibrium driven by a noise. Perturbing the state by the insertion of an operator, and looking at its approach back to equilibrium is just what is computed in correlation functions with operator insertions at different times. General properties of the approach to equilibrium can then be used to show that correlation functions of operators at different times fall off with the time difference. In fact for a theory with a mass gap they decay exponentially and with no mass gap they decay with a power law.

## 5. Conclusions and further discussion

In this paper we have argued that the gauge invariant Schwinger-Dyson operator has a natural dual in the dual supergravity theory as the supergravity hamiltonian conjugate to the radial foliation of the AdS. It will become the string field theory hamiltonian away from the semi-classical limit. Using this we have shown that there is a natural (though far from simple) class of observables that mimics the existence of an extra large dimension. Properties of these observables were shown to coincide with general expectations, but the construction enables one to understand the connection between the gauge theory and the supergravity in a deeper way.

One interesting question that can be addressed with this identification is the issue of Minkowski holography. In our formalism, this question turns into: what gauge theory action has a Schwinger-Dyson operator that is equal to the Wheeler-DeWitt operator for a vanishing cosmological constant?

## Acknowledgments

We would like to thank A. Polyakov and H. Verlinde for discussions. G.L. would like to thank W. Fischler, I. Klebanov, S. Shenker, L. Susskind and E. Verlinde for helpful conversations. V.P. is grateful to B. Durand and L. Durand for their hospitality at the University of Wisconsin, Madison. This work was supported in part by NSF grant PHY98-02484.

## References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[5] D. Kabat and G. Lifschytz, *Tachyons and black hole horizons in gauge theory*, *J. High Energy Phys.* **12** (1998) 002 [`hep-th/9806214`];
*Gauge theory origins of supergravity causal structure*, *J. High Energy Phys.* **05** (1999) 005 [`hep-th/9902073`]
S.R. Das, *Brane waves, Yang-Mills theories and causality*, *J. High Energy Phys.* **02** (1999) 012 [`hep-th/9901004`];
G.T. Horowitz and N. Itzhaki, *Black holes, shock waves and causality in the AdS/CFT correspondence*, *J. High Energy Phys.* **02** (1999) 010 [`hep-th/9901012`].

[6] T. Banks, M.R. Douglas, G.T. Horowitz and E. Martinec, *AdS dynamics from conformal field theory*, `hep-th/9808016`;
V. Balasubramanian, P. Kraus, A. Lawrence and S. Trivedi, *Phys. Rev.* **D 59** (1999) 104021.

[7] G. 't Hooft, in *Salamfestschrift: a collection of talks*, World Scientific Series in 20th century physics, v. 4, A. Ali et al. eds.. World Scientific 1993; in Proc. Symp. *The Oskar Klein centenary*, ed. U. Lindström, World Scientific 1995;
L. Susskind, *J. Math. Phys.* **36** (1995) 6377.

[8] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, `hep-th/9805114`.

[9] N. Ishibashi and H. Kawai, *String field theory of noncritical strings*, *Phys. Lett.* **B 314** (1993) 190 [`hep-th/9307045`];
M. Fukuma, N. Ishibashi, H. Kawai and M. Ninomiya, *Two-dimensional quantum gravity in temporal gauge*, *Nucl. Phys.* **B 427** (1994) 139 [`hep-th/9312175`];
M. Ikehara et al., *String field theory in the temporal gauge*, *Phys. Rev.* **D 50** (1994) 7467 [`hep-th/9406207`];
N. Ishibashi and H. Kawai, *String field theory of $c \leq 1$ noncritical strings*, *Phys. Lett.* **B 322** (1994) 67 [`hep-th/9312047`]; *A background independent formulation of noncritical string theory*, *Phys. Lett.* **B 352** (1995) 75 [`hep-th/9503134`].

[10] A. Jevicki and J. Rodrigues, *Loop space hamiltonians and field theory of noncritical strings*, *Nucl. Phys.* **B 421** (1994) 278 [`hep-th/9312118`];
see also S. Das and A. Jevicki, *String field theory and physical interpretation of $D = 1$ strings*, *Mod. Phys. Lett.* **A 5** (1990) 1639;
G. Moore, N. Seiberg and M. Staudacher, *From loops to states in 2D quantum gravity*, *Nucl. Phys.* **B 362** (1991) 665.

[11] G. Parisi and Y.-S. Wu, *Perturbation theory without gauge fixing*, *Sci. Sin.* **24** (1981) 484.

[12] G. Marchesini, *A comment on the stochastic quantization: the loop equation of gauge theory as the equilibrium condition*, *Nucl. Phys.* **B 191** (1981) 214; *Loop dynamics for gauge theories: a numerical algorithm*, *Nucl. Phys.* **B 239** (1984) 135.

[13] V. Periwal, *String field theory hamiltonians from Yang-Mills theories*, hep-th/9906052 to appear on *Phys. Rev.* **D**.

[14] A. Polyakov, *String theory and quark confinement*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 1 [hep-th/9711002]; *The wall of the cave*, *Int. J. Mod. Phys.* **A 14** (1999) 645 [hep-th/9809057].

[15] G.F. D. Angelis, D. de Falco and F. Guerra, *Lattice gauge models in the strong coupling regime*, *Lett. Nuovo Cim.* **19** (1977) 55;
F. Guerra, R. Marra and G. Immirzi, *Strong coupling expansion for lattice Yang-Mills fields*, *Lett. Nuovo Cim.* **23** (1978) 237;
J.-L. Gervais and A. Neveu, *The quantum dual string wave functional in Yang-Mills theories*, *Phys. Lett.* **B 80** (1979) 255;
Y. Nambu, *QCD and the string model*, *Phys. Lett.* **B 80** (1979) 372;
E. Corrigan and B. Hasslacher, *A functional equation for exponential loop integrals in gauge theories*, *Phys. Lett.* **B 81** (1979) 181;
A.M. Polyakov, *String representations and hidden symmetries for gauge fields*, *Phys. Lett.* **B 82** (1979) 247;
L. Durand and E. Mendel, *Functional equations for path dependent phase factors in Yang-Mills theories*, *Phys. Lett.* **B 85** (1979) 241;
D. Förster, *Yang-Mills theory: a string theory in disguise*, *Phys. Lett.* **B 87** (1979) 87;
T. Eguchi, *Strings in* U($N$) *lattice gauge theory*, *Phys. Lett.* **B 87** (1979) 91;
Y.M. Makeenko and A.A. Migdal, *Exact equation for the loop average in multicolor QCD*, *Phys. Lett.* **B 88** (1979) 135;
A. Jevicki and B. Sakita, *Collective field approach to the large-$N$ limit: euclidean field theories*, *Nucl. Phys.* **B 185** (1981) 89.

[16] G. Lifschytz and V. Periwal, *Dynamical truncation of the string spectrum at finite $n$*, hep-th/9909152.

[17] S. Hirano, *Exact renormalization group and loop equation*, hep-th/9910256;
M. Li, *A note on relation between holographic rg equation and Polchinski's RG equation*, hep-th/0001193;
C. van de Bruck, *On gravity, holography and the quantum*, gr-qc/0001048.

[18] P.H. Damgaard and H. Huffel, *Stochastic quantization*, *Phys. Rep.* **152** (1987) 227;
J. Zinn-Justin, *Quantum field theory and critical phenomena*, Oxford University Press 1996.

[19] M. Ikehara, N. Ishibashi, H. Kawai, T. Mogami, R. Nakayama and N. Sasakura, *A note on string field theory in the temporal gauge*, *Prog. Theor. Phys. Suppl.* **118** (1995) 241.

[20] M.B. Halpern, *Universal continuum regularization of quantum field theory*, *Prog. Theor. Phys. Suppl.* **111** (1993) 163.

[21] S.-J. Rey and J. Yee, *Macroscopic strings as heavy quarks in large-$N$ gauge theory and anti-de Sitter supergravity*, `hep-th/9803001`;
J. Maldacena, *Wilson loops in large-$N$ field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [`hep-th/9803002`].

[22] J. Zinn-Justin, *Renormalization and stochastic quantization*, *Nucl. Phys.* **B 275** (1986) 135;
*Renormalization and stochastic dynamical equations*, *Prog. Theor. Phys. Suppl.* **111** (1993) 185.

[23] J. de Boer, E. Verlinde and H. Verlinde, *On the holographic renormalization group*, `hep-th/9912012`.

[24] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, `hep-th/0002230`.

[25] A. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, *Phys. Rev.* **D 59** (1999) 065011 [`hep-th/9809022`].